Mathematics
Faculté des sciences d'Orsay

Sheet 2

Groups and geometry
M2 AAG 2024–2025

Around the exponential map

Exercise 1 Let G be a real or complex Lie group and e its identity element.
Show there exists U a neighbourhood of identity in G such that if H is a subgroup of G contained in U, then $H = \{e\}$.

Exercise 2 Let G and H be Lie groups. Assume G is connected.

1. Consider φ et ψ two Lie group morphisms from G to H. Assume there exists $g_0 \in G$ such that $\varphi(g_0) = \psi(g_0)$ et $T_{g_0}\varphi = T_{g_0}\psi$. Show that $T_e\varphi = T_e\psi$, then that φ and ψ coincide on a neighbourhood of g_0. Finally, conclude that $\varphi = \psi$.
2. Let φ be a C^∞ diffeomorphism of G, such that for any left-invariant vector field X, we have $\varphi^*X = X$. Show that there exists $g_0 \in G$ such that $\varphi = L_{g_0}$.

Adjoint representation and invariant differential forms

Exercise 3 Let G be a Lie group of dimension $n \geq 1$, with Lie algebra $\mathfrak{g}$.

1. Show that the differential s-forms ω on G that are left-invariant, i.e. such that $L_g^*(\omega) = \omega$ for any $g \in G$, form a vector space in natural bijection with $(\Lambda^s(\mathfrak{g}))^*$.
2. Let ω be a left-invariant s-form on G. Show that for any $g \in G$, $R_g^*(\omega)$ is left-invariant. What element in $\Lambda^s(\mathfrak{g})$ does it correspond to?
3. Show that if G is compact and connected, then any left-invariant or right-invariant n-form is automatically bi-invariant.
4. What about for the noncompact groups $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{Aff}^+(\mathbb{R})$?

On compact connected complex Lie groups

Exercise 4 Let G be a connected compact complex Lie group.
Let V be a finite-dimensional complex vector space and $\rho : G \to \mathrm{GL}(V)$ be a complex Lie group morphism.

1. Show that $\rho(G) = \{\mathrm{Id}\}$.
   (*Indication.* A bounded holomorphic map from $\mathbb{C}$ to $\mathbb{C}$ is constant.)
2. Deduce that any connected compact complex Lie group is commutative.