Sheet 3

Bracket between vector fields, and the exponential map

Exercise 1 Let X be a left-invariant vector field on a Lie group G.
Compute the flow φ_t of X.

Exercise 2 Let X, Y be two vector fields on a manifold M and φ_t, ψ_t the local flows of X and Y.

1. Show that $[X,Y] = \frac{d}{dt}\big|_{t=0}(\varphi_t)^*(Y)$.
2. Show that, for all $x \in M$, $\frac{d}{dt}\big|_{t=0}\psi_t \circ \varphi_t \circ \psi_{-t} \circ \varphi_{-t}(x) = 0$.
3. Show that, for all $x \in M$, $[X,Y](x) = \frac{d}{dt}\big|_{t=0^+}\psi_{\sqrt{t}} \circ \varphi_{\sqrt{t}} \circ \psi_{-\sqrt{t}} \circ \varphi_{-\sqrt{t}}(x)$.

Let X, Y be two elements in the Lie algebra $\mathfrak{g}$ of a Lie group G, and $\exp : \mathfrak{g} \to G$ the exponential map of G.

4. Show that $\frac{d}{dt}\big|_{t=0} \exp(tX)\exp(tY)\exp(-tX)\exp(-tY) = 0$.
5. Show that $[X,Y] = \frac{d}{dt}\big|_{t=0^+} \exp(\sqrt{t}X)\exp(\sqrt{t}Y)\exp(-\sqrt{t}X)\exp(-\sqrt{t}Y)$.

Universal cover of the group $\mathrm{SL}_2(\mathbb{R})$

Exercise 3 Denote by $\mathrm{SL}_2(\mathbb{R})$ the group of 2×2 matrices of determinant $+1$ with real entries.

1. Show that any one-parameter subgroup $(g^t)_{t\in\mathbb{R}}$ of $\mathrm{SL}_2(\mathbb{R})$ is conjugate in $\mathrm{SL}_2(\mathbb{R})$ to one of the following three subgroups:
$$\left(s^t = \begin{pmatrix} e^{at} & 0 \\ 0 & e^{-at} \end{pmatrix}\right)_{t\in\mathbb{R}}, \quad \left(u^t = \begin{pmatrix} 1 & at \\ 0 & 1 \end{pmatrix}\right)_{t\in\mathbb{R}}, \quad \left(r^t = \begin{pmatrix} \cos at & -\sin at \\ \sin at & \cos at \end{pmatrix}\right)_{t\in\mathbb{R}}.$$
If a is nonzero, $(g^t)_{t\in\mathbb{R}}$ is called *hyperbolic* in the first case, *parabolic* in the second case, *elliptic* otherwise.
2. If $g^t = \exp(tZ)$ with $Z \in \mathfrak{sl}_2(\mathbb{R})$, relate the hyperbolic, parabolic or elliptic nature of (g^t) to the sign of $B(Z,Z)$, where B is the Killing form of $\mathfrak{sl}_2(\mathbb{R})$.
3. Is the map $\exp : \mathfrak{sl}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$ surjective?

Denote by $\pi : \widetilde{\mathrm{SL}_2(\mathbb{R})} \to \mathrm{SL}_2(\mathbb{R})$ the universal cover of $\mathrm{SL}_2(\mathbb{R})$, and by $\widetilde{\exp}$ the exponential map from $\mathfrak{sl}_2(\mathbb{R})$ to $\widetilde{\mathrm{SL}_2(\mathbb{R})}$.

4. What is the relation between $\exp$, $\widetilde{\exp}$ and π?
5. Describe the centre of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$.
6. Show that if $B(Z,Z) < 0$, then $(\widetilde{\exp}(tZ))_{t\in\mathbb{R}}$ is an embedded Lie subgroup of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, isomorphic to $\mathbb{R}$ and containing the centre of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$.
7. Are there any connected compact subgroups besides $\{e\}$ in $\widetilde{\mathrm{SL}_2(\mathbb{R})}$?