Mathematics
Faculté des sciences d'Orsay

Sheet 4

Groups and geometry
M2 AAG 2024–2025

Exercise 1 Classify up to isomorphism
- all connected Lie groups with $\mathfrak{aff}(\mathbf{R})$ as their Lie algebra;
- all connected Lie groups with $\mathfrak{heis}(3)$ as their Lie algebra.

Exercise 2 Let G be a Lie group with Lie algebra $\mathfrak{g}$.

1. Let H be a closed subgroup of G.
   Show that if H is not discrete, it contains a nontrivial one-parameter subgroup.
2. Denote by $\mathfrak{z}(\mathfrak{g})$ and $Z(G)$ the centres of the Lie algebra $\mathfrak{g}$ and of the Lie group G.
   Assuming G is connected, show that $Z(G)$ is discrete if and only if $\mathfrak{z}(\mathfrak{g}) = \{0\}$.
3. Give an example of a connected Lie group G such that $\mathfrak{z}(\mathfrak{g}) = \{0\}$ but $Z(G) \neq \{e\}$.

Exercise 3 Let V be an $\mathbf{R}$-vector space of finite dimension and $\omega : V \times V \to \mathbf{R}$ an anti-symmetric bilinear form. Consider the operation $*$ on $V \times \mathbf{R}$ defined by

$$\forall v, v' \in V \ \forall t, t' \in \mathbf{R} \quad (v,t) * (v',t') = \left(v + v', t + t' + \frac{1}{2}\omega(v, v')\right)$$

1. Show that $(V \times \mathbf{R}, *)$ is a Lie group, which we will denote by G.
2. Describe the Lie algebra $\mathfrak{g}$ of G, and show it is nilpotent.
3. Show that $\exp_G : \mathfrak{g} \to G$ is a diffeomorphism.

Exercise 4 Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbf{R}$ such that $\mathfrak{z}(\mathfrak{g}) = \{0\}$.
Show there exists a connected Lie group G such that $Z(G) = \{e\}$,
whose Lie algebra is isomorphic to $\mathfrak{g}$.
Show that such a Lie group is unique up to isomorphism.

Exercise 5 Let $n \geq 1$ be an integer.
Show any compact subgroup of $\mathrm{GL}(n, \mathbf{R})$ is conjugate to a subgroup of $\mathrm{O}(n, \mathbf{R})$.

Exercise 6 Recall that the Heisenberg group, Heis(3), consists in upper triangular matrices in $\mathcal{M}_3(\mathbf{R})$ with all diagonal elements equal to 1.
Let Γ be the subgroup generated by $\left(\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ and $Z = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) \in \mathfrak{heis}(3)$.

1. Let $\varphi : \mathfrak{heis}(3) \to \mathfrak{sl}_n(\mathbf{R})$ be a Lie algebra morphism. Show that $\varphi(Z)$ is nilpotent.
2. Deduce that $\{\exp(t\varphi(Z)) \,|\, t \in \mathbf{R}\}$ is either trivial or unbounded.

Let $\rho : \mathrm{Heis}(3)/\Gamma \to \mathrm{GL}_n(\mathbf{R})$ be a Lie group morphism.

3. Show that the one-parameter sugbroup $(\exp(t d_e\rho(Z)))_t$ is compact.
4. Deduce that ρ is not injective.
   In other words, $\mathrm{Heis}(3)/\Gamma$ has no faithful linear representation.