Sheet 5

Coadjoint orbits

Exercise 1 Fix a smooth action of a compact real Lie group G on a manifold M.
To each X in $\mathfrak{g}$, associate the vector field X_M on M defined by $X_M(x) = \frac{d}{dt}_{|t=0} \exp(tX)x$.

1. Show that, for any X, Y in $\mathfrak{g}$, $[X_M, Y_M] = -[X, Y]_M$.
2. Given $x \in M$, denote by $\mathfrak{g}_x$ the Lie algebra of the stabiliser G_x.
   Show that $X \mapsto X_M(x)$ induces an isomorphism between $\mathfrak{g}/\mathfrak{g}_x$ and $T_x(G \cdot x)$.

Call adjoint representation of G the representation $G \to \mathrm{GL}(\mathfrak{g})$ that maps g to $\mathrm{Ad}(g)$, and coadjoint representation of G the representation $G \to \mathrm{GL}(\mathfrak{g}^*)$ that maps g to ${}^t(\mathrm{Ad}(g^{-1}))$.

3. Show that, for all X, Y in $\mathfrak{g}$, $X_{\mathfrak{g}}(Y) = [X, Y]$.
4. Show that, for all X in $\mathfrak{g}$ and all λ in $\mathfrak{g}^*$, $X_{\mathfrak{g}^*}(\lambda) = -\,{}^t(\mathrm{ad}\, X)(\lambda)$.
5. Fix λ in $\mathfrak{g}^*$. Show that the bilinear form $\underline{\omega}$ defined on $\mathfrak{g}$ by $\underline{\omega}(X, Y) = \lambda([X, Y])$ passes to the quotient to a nondegenerate bilinear form on $\mathfrak{g}/\mathfrak{g}_\lambda$.
6. Show that the G-invariant bilinear form ω induced by $\underline{\omega}$ on G/G_λ is symplectic, i.e. it is a differential 2-form that is nondegenerate at each point and closed.
7. Describe the orbits of the coadjoint action of $\mathrm{SO}_3(\mathbb{R})$.

Actions of $\mathrm{SL}_2(\mathbb{R})$ and representations of $\mathfrak{sl}_2(\mathbb{R})$

Exercise 2 We are interested in the action of $\mathrm{SL}_2(\mathbb{R})$ on (very) low-dimensional manifolds.

1. Let M be a connected manifold of dimension 1 admitting a smooth and transitive action of the group $\mathrm{SL}_2(\mathbb{R})$.
   (a) Show that such an action implies the existence of a sub-Lie algebra $\mathfrak{h}$ of $\mathfrak{sl}_2(\mathbb{R})$ isomorphic to $\mathfrak{aff}(\mathbb{R})$.
   (b) By studying the eigenvalues of $\mathrm{ad}(X)$ for $X \in \mathfrak{h}$, show that $\mathfrak{h}$ can be identified to the subalgebra of $\mathfrak{sl}_2(\mathbb{R})$ generated by $E = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.
   (c) Deduce that M is diffeomorphic to the circle S^1.
2. Deduce that any smooth action of $\mathrm{SL}_2(\mathbb{R})$ on the circle is either trivial or transitive.
3. Show that there is no other nontrivial smooth action of $\mathrm{SL}_2(\mathbb{R})$ on the real line.
4. Consider a smooth action of $\mathrm{SL}_2(\mathbb{R})$ on the sphere S^2. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$.
   (a) Using Exercise 1, show there exist points x, y in S^2 such that $H \in \mathfrak{g}_x$ and $E \in \mathfrak{g}_y$.
   One can use the fact that any smooth vector field on S^2 vanishes somewhere.
   (b) Deduce that the action under consideration cannot be transitive.