Mathematics
Faculté des sciences d'Orsay

Sheet 6

Groups and geometry
M2 AAG 2024–2025

Exercise 1 *Canonical connection of a riemannian submanifold.*

1. Using the following coordinates for the sphere S^2:
$$(\theta, \varphi) \mapsto (\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta)$$
compute
$$D_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\theta}, \qquad D_{\frac{\partial}{\partial\varphi}}\frac{\partial}{\partial\theta}, \qquad D_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\varphi}, \qquad D_{\frac{\partial}{\partial\varphi}}\frac{\partial}{\partial\varphi}.$$
2. Parametrize the torus T^2 from $S^1 \times S^1$ in two ways:
$$\varphi_1(\theta, \varphi) = (\exp(i\theta), \exp(i\varphi))$$
$$\varphi_2(\theta, \varphi) = ((2+\cos\theta)\cos\varphi, (2+\cos\theta)\sin\varphi, \sin\theta)$$
Compare the values of $[\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\varphi}]$ for T^2 with its structure of riemannian submanifold of $\mathbb{R}^4$ given by φ_1 and its structure of riemannian submanifold of $\mathbb{R}^3$ given by φ_2.

Exercise 2 *Parallel transport on a cone and on a sphere.*

1. In $\mathbb{R}^3$ with cartesian coordinates (x, y, z), consider: a half-line from the origin in the xz plane at angle α with Oz; the revolution cone C of axis Oz that it generates; a curve $c : [a, b] \to C$ looping around the cone vertex; a parallel vector field X along c.
Compute the angle between $X(a)$ and $X(b)$.
2. Same question for a parallel vector field along a small circle on the sphere S^2.

Exercise 3 *Geodesics.*

1. Geodesics on $\mathbb{R}^n$ are straight lines parametrized at constant velocity.
2. Geodesics on a riemannian n-manifold $M \subset \mathbb{R}^{n+p}$ are the curves with normal acceleration vector field (i.e. the field of acceleration vectors is everywhere normal to M).
3. Geodesics on the sphere (S^n, can) are great circles parametrized at constant velocity.

Exercise 4 *More geodesics.*

1. Geodesics on the standard flat n-torus $\mathbb{R}^n/\mathbb{Z}^n$.
2. Geodesics on other flat n-tori $\mathbb{R}^n/\Lambda$.
3. Geodesics on the standard flat Klein bottle.
4. Geodesics on other flat Klein bottles.