Exercise 1. Let G be a (real) Lie group with Lie algebra $\mathfrak{g}$.

1. Prove there is a unique connection ∇ on TG such that, for any left-invariant vector fields X and Y on G, $\nabla_X Y = \frac{1}{2}[X, Y]$.

2. Prove that ∇ is torsion free.

3. Compute $R(X, Y)Z$ for left-invariant vector fields X, Y and Z.

4. Prove that, for every X in $\mathfrak{g}$ and every $g \in G$, $t \mapsto g \exp_G(tX)$ is a geodesic for ∇. Prove that all geodesics have this shape.

5. In this question, we assume G is equipped with a Riemannian metric h that is invariant under all left and right translations (we say h is bi-invariant). We denote by h_1 the corresponding inner product on $\mathfrak{g}$.

   (a) Prove that, for every $g \in G$, h_1 is $\mathrm{Ad}(g)$-invariant.

   (b) Prove that, for every $X \in \mathfrak{g}$, $\mathrm{ad}(X)$ is antisymmetric on $\mathfrak{g}$ with respect to h_1.

   (c) Using the method used in class to prove uniqueness of the Levi-Civita connection, compute that Levi-Civita connection acting on left-invariant vector fields. Describe Riemannian geodesics.

   (d) For every X and Y in $\mathfrak{g}$ with $\|X\| = \|Y\| = 1$ and $X \perp Y$, compute the sectional curvature of $\mathrm{Span}(X, Y)$.

6. We now assume G is compact.

   (a) Using a right-invariant volume form on G, prove that $\mathfrak{g}$ admits an inner product invariant under the adjoint action of G on $\mathfrak{g}$.

   (b) Prove that G admit a bi-invariant metric.

   (c) Prove $\exp_G$ is surjective.