Exercise 1. Let G be a (real) Lie group with Lie algebra $\mathfrak{g}$.

1. Prove there is a unique connection ∇ on TG such that, for any left-invariant vector fields X and Y on G, $\nabla_X Y = \frac{1}{2}[X, Y]$.

   Solution. We fix once and for all a basis $e_1, \dots, e_n$ of $\mathfrak{g}$ and use the same notation for the corresponding left-invariant vector fields on G. Any vector field X on G can be decomposed as $\sum_i X^i e_i$ where each X^i is a function from G to $\mathbb{R}$. The vector field X is left-invariant if and only if every X_i is a constant function. Indeed constant function clearly give rise to an invariant X. Conversely if X is invariant then X coincides at 1, hence everywhere, with the left-invariant $\sum_i X^i(1)e_i$ that has constant component functions.

   We first prove uniqueness of ∇. Assume ∇ is such a connection. Then for every vector fields X and Y, we have

   $$\begin{aligned}\nabla_X Y &= \nabla_{\sum_i X^i e_i}\left(\sum_j Y^j e_j\right)\\ &= \sum_i X^i \nabla_{e_i} \sum_j \left(Y^j e_j\right) \quad \text{since } \nabla \text{ is a connection}\\ &= \sum_i X^i \sum_j \left((e_i Y^j)e_j + Y^j \nabla_{e_i} e_j\right) \quad \text{since } \nabla \text{ is a connection}\\ &= \sum_i X^i \sum_j \left((e_i Y^j)e_j + Y^j \frac{1}{2}[e_i, e_j]\right) \quad \text{by assumption}\end{aligned}$$

   The end result does not depend on ∇ so we proved uniqueness.

   For existence, we define $\nabla_X Y$ by the above formula. We first need to prove that we get a connection. The linearity over $\mathcal{C}^\infty(G)$ for the X slot is clear, and same for linearity over $\mathbb{R}$ for the Y slot. We need to check the product rule. In $\nabla_X(fY)$ the only extra piece compared to $f\nabla_X Y$ comes from $e_i(fY^j) = (e_i f)Y^j + f e_i Y^j$ so we get

   $$\begin{aligned}\nabla_X(fY) &= f\nabla_X Y + \sum_i X^i \sum_j ((e_i f)Y^j)e_j\\ &= f\nabla_X Y + \sum_j (\sum_i X^i(e_i f))Y^j e_j\\ &= f\nabla_X Y + \sum_j (Xf)Y^j e_j\\ &= f\nabla_X Y + (Xf)Y\end{aligned}$$

It remains only to show that ∇ satisfies the required formula for left invariant vector fields. This follows from the fact that those vector fields have constant components on the e_i basis so there are no $e_i Y^j$ terms for them.

2. Prove that ∇ is torsion free.

   Solution. Fix g in G and fix u and v in $T_g G$. We want to prove $T(u, v) = 0$. We know that, for every vector fields X and Y taking values u and v at g, we have $T(u, v) = \nabla_X Y - \nabla_Y X - [X, Y]$. We can choose such X and Y that are left-invariant. Indeed one can use left-translation to trivialize TG and extend u and v to constant vector fields. More explicitly, one can simply define $X = (g' \mapsto T_g L_{g' g^{-1}}(u))$ and use the same idea for Y.

   The announced vanishing is then a direct consequence of the defining property of ∇.

3. Compute $R(X, Y)Z$ for left-invariant vector fields X, Y and Z.

   Solution. We use the definition and the Jacobi identity for the Lie bracket.

$$\begin{aligned} R(X,Y)Z &= \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z \\ &= \frac{1}{4}[X,[Y,Z]] - \frac{1}{4}[Y,[X,Z]] - \frac{1}{2}[[X,Y],Z] \\ &= \frac{1}{4}([[X,Y],Z] + [Y,[X,Z]]) - \frac{1}{4}[Y,[X,Z]] - \frac{1}{2}[[X,Y],Z] \\ &= -\frac{1}{4}[[X,Y],Z]. \end{aligned}$$

4. Prove that, for every X in $\mathfrak{g}$ and every $g \in G$, $t \mapsto g \exp_G(tX)$ is a geodesic for ∇. Prove that all geodesics have this shape.

   Solution. Fix g in G and X in $\mathfrak{g}$. Set $\gamma = (t \mapsto g \exp_G(tX))$. We want to prove that γ is a geodesic for ∇. The key point from the Lie group part of the course is the formula, for every t, $\gamma(t) = \varphi_X^t(g)$ where φ_X is the flow of X (seen as a left-invariant vector field on G). Hence, for every t, $\gamma'(t) = X_{\gamma(t)}$ (one can also rederive this directly).

   In particular γ' seen as a vector field along γ actually comes from an ambiant vector field. The fundamental property of induced connections then gives $\nabla_{\gamma'}\gamma' = \nabla_{\gamma'}X = \nabla_X X$. But X is left-invariant so $\nabla_X X = \frac{1}{2}[X, X] = 0$ and γ is indeed a geodesic.

We now claim that every geodesic has this shape. Indeed we have geodesics with this shape for every possible initial condition. Fix g in G and v in $T_g G$. Let X be the unique left-invariant vector field with value v at g. We see X as an element of $\mathfrak{g}$. Then $t \mapsto g \exp_G(tX)$ starts at g with velocity $T_1 L_g(X) = X_g = v$.

5. In this question, we assume G is equipped with a Riemannian metric h that is invariant under all left and right translations (we say h is bi-invariant). We denote by h_1 the corresponding inner product on $\mathfrak{g}$.

   (a) Prove that, for every $g \in G$, h_1 is $\mathrm{Ad}(g)$-invariant.

   Solution. By assumption, h is both left and right invariant so, for every g, $L_g^* h = R_g^* h = h$. And $c_g = L_g \circ R_g$ so $c_g^* h = h$. This is true at every point in G, and gives the announced result $\mathrm{Ad}(g)^* h_1 = h_1$ at 1.

   Remark: there are two closely related pull-back operations at play here. The purely algebraic one takes an inner product b on a vector space F and a linear map $\varphi : E \to F$ and builds $\varphi^* b$ on E sending (e, e') to $b(\varphi(e), \varphi(e'))$. If φ is an isomorphism then $\varphi^* b$ is also an inner product. Let's use the temporary notation $\varphi^\star b$ for this algebraic version. Now consider a smooth map $f : M \to N$ and a Riemannian metric h on N. We can pull back h to get $f^* h$. At every $m \in M$, $f^* h$ is the bilinear form $(T_m f)^\star h_{f(m)}$ on $T_m M$. If f is (local) diffeomorphism, $f^* h$ is a Riemannian metric. When $M = N = G$ and $f = c_g$ and $m = 1$ we have $(c_g^* h)_1 = \mathrm{Ad}(g)^\star h_1$.

   (b) Prove that, for every $X \in \mathfrak{g}$, $\mathrm{ad}(X)$ is antisymmetric on $\mathfrak{g}$ with respect to h_1.

   Solution. The previous question ensures the Lie group morphism $\mathrm{Ad} : G \to \mathrm{End}(\mathfrak{g})$ lands into the orthogonal group of h_1. Its derivative at 1, which is ad, goes from $\mathfrak{g}$ to the Lie algebra of the orthogonal group which is the space of h_1–antisymmetric maps.

   (c) Using the method used in class to prove uniqueness of the Levi-Civita connection, compute that Levi-Civita connection acting on left-invariant vector fields. Describe Riemannian geodesics.

   Solution. Consider three left-invariant vector fields X, Y and Z. As in the proof of the uniqueness of the Levi-Civita connection, we can write the compatibility condition for all circular permutations of X, Y and Z and use the torsion free condition and a suitable combination of the three compatibility condition to keep only $\nabla_X Y$ as the only covariant derivative. What is specific to our current situation is that all inner products of those vector fields are constant since they are left-invariant functions on G. So we get:

$$2\langle \nabla_X Y, Z\rangle = \langle [X,Y], Z\rangle + \langle Y, [X,Z]\rangle + \langle X, [Y,Z]\rangle$$

Next we can use the previous question which ensures that ad is anti-symmetric to rewrite this as

$$2\langle \nabla_X Y, Z\rangle = \langle [X,Y], Z\rangle - \langle [X,Y], Z\rangle - \langle [Y,X], Z\rangle$$

which then gives

$$\langle \nabla_X Y, Z\rangle = \langle \frac{1}{2}[X,Y], Z\rangle$$

Since this is true for every left-invariant Z and, at each point, every vector is the value of a left-invariant vector field, we get the expected formula for $\nabla_X Y$. According to Question 4, Riemannian geodesics are exactly the left-translates of one-parameter subgroups.

(d) For every X and Y in $\mathfrak{g}$ with $\|X\| = \|Y\| = 1$ and $X \perp Y$, compute the sectional curvature of $\mathrm{Span}(X,Y)$.

Solution. We use the formula derived for R and antisymmetry of $\mathrm{ad}(Y)$.

$$\begin{aligned} K &= R(X,Y,Y,X) \\ &= \langle -\frac{1}{4}[[X,Y],Y], X\rangle \\ &= \frac{1}{4}\langle \mathrm{ad}(Y)([X,Y]), X\rangle \\ &= -\frac{1}{4}\langle [X,Y], \mathrm{ad}(Y)(X)\rangle \\ &= \frac{1}{4}\|[X,Y]\|^2 \end{aligned}$$

6. We now assume G is compact.

(a) Using a right-invariant volume form on G, prove that $\mathfrak{g}$ admits an inner product invariant under the adjoint action of G on $\mathfrak{g}$.

Solution. Let Ω be a right-invariant volume form on G (it can be constructed by choosing any volume form on the Lie algebra and pushing it by all right translations R_g).

We start with any inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ and average it to

$$h_1 := \int_{g\in G} \operatorname{Ad}(g)^*\langle\cdot,\cdot\rangle\Omega$$

Let us check this is Ad-invariant. Fix $g_0 \in G$.

$$\begin{aligned}
\operatorname{Ad}(g_0)^* h_1 &= \operatorname{Ad}(g_0)^* \int_{g\in G} \operatorname{Ad}(g)^*\langle\cdot,\cdot\rangle\Omega \\
&= \int_{g\in G} \operatorname{Ad}(g_0)^* \operatorname{Ad}(g)^*\langle\cdot,\cdot\rangle\Omega \\
&= \int_{g\in G} \operatorname{Ad}(gg_0)^*\langle\cdot,\cdot\rangle\Omega \\
&= \int_{g\in G} ((\operatorname{Ad}\circ R_{g_0})(g))^*\langle\cdot,\cdot\rangle R_{g_0}^*\Omega \\
&= \int_{g\in G} (\operatorname{Ad}(g)^*\langle\cdot,\cdot\rangle
\end{aligned}$$

The second-to-last equality uses that Ω is right-invariant and the last equality is the change of variable theorem for integrals of differential forms: for every top-degree form α on G with compact support and every measurable $A \subseteq G$, $\int_A R_g^*\alpha = \int_{R_g(A)} \alpha$.

(b) Prove that G admit a bi-invariant metric.

Solution. The previous question gives us an inner product h_1 on $\mathfrak{g}$ which is Ad-invariant. We then use left-translations to turn it into a metric h which is left-invariant by construction. We claim that is it also right-invariant. In order to prove this, fix $g \in G$. We want to prove $R_g^* h = h$. Both sides are left-invariant metrics on G (using that every left-translation commutes with every right translation and h is left-invariant). So it suffices two prove they coincide at 1. And $R_g^* h = c_g^* h$ since h is left-invariant so they desired equality is exactly $\operatorname{Ad}(g)^* h_1 = h_1$ (see the discussion in the solution to Question 5a).

(c) Prove $\exp_G$ is surjective.

Solution. Fix g in G. We want to prove g is in the image of $\exp_G$. Equip G with a bi-invariant metric given by the previous question. Since G is compact, it is metrically complete and Hopf-Rinow gives a (minimizing) geodesic γ from 1 to g. By Question 5c, γ is a left-translate of a one-parameter subgroup: $\gamma = (t \mapsto g_0 \exp_G(tX))$ for some g_0 in G and some X in $\mathfrak{g}$. Since $\gamma(0) = 1$, $g_0 = 1$ and g is indeed in the image of $\exp_G$.